Announcements

1) HW #3 will be due next week - there is now a hint for the "space-filling curve" question.

Back to Weierstrass Approximation

We showed that if

$f: [0,1] \to \mathbb{R}$ is continuous,

$f(0) = f(1) = 0$, then

$\exists$ a sequence of polynomials

$(P_n)_{n=1}^{\infty}$, $P_n \to f$ uniformly

on $[0,1]$.

Now suppose $f : [a,b] \to \mathbb{R}$ is continuous.

Consider the map

$$\varphi : [a,b] \to [0,1]$$

$$\varphi(t) = \frac{(t-a) + (t-b) - (a-b)}{2(b-a)}$$

linear in t

So then if P is any polynomial,

$P(\varphi(t))$ is also a polynomial.

Consider

$g : [0,1] \to \mathbb{R}$

$g(t) = f(\varphi^{-1}(t))$

$g(0) = f(\varphi^{-1}(0)) = f(a)$

$g(1) = f(\varphi^{-1}(1)) = f(b)$

Observe

$$\varphi^{-1}(t) = bt + (1-t)a.$$

Now let

$$h : [0,1] \to \mathbb{R}$$

$$h(t) = g(t) - t\,(g(1) - g(0)) - g(0)$$

$$h(0) = g(0) - g(0) = 0$$

$$h(1) = g(1) - g(1) + g(0) - g(0) = 0$$

Apply Weierstrass Approximation to h.

Then $\exists$ a sequence of polynomials $(Q_n)_{n=1}^{\infty}$, $Q_n \to h$ uniformly on $[0,1]$.

$$\Rightarrow Q_n(t) + t\big(g(1)-g(0)\big) + g(0) = S_n(t)$$ is a sequence of polynomials that converge uniformly to g.

Now we let

$$P_n : [a,b] \to \mathbb{R}$$

$$P_n(t) = S_n(\varphi(t)).$$

Then $P_n \to f$ uniformly on $[a,b]$ and is a sequence of polynomials, since $f = g \circ \varphi^{-1}$.

Improper Integration

We define, for $a \in \mathbb{R}$

and $f : [a, \infty) \to \mathbb{R}$,

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

provided the limit exists!

If $f: (-\infty, a] \to \mathbb{R}$,

define

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{a} f(x)\,dx$$

provided the limit exists.

We finally define

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{a}^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx$$

where $a \in \mathbb{R}$ and provided

both improper integrals exist.

Logarithms

Let $x \geq 1$ be a real number. Observe that

$f(t) = \frac{1}{t}$ is continuous

on $[1, x]$ $\forall$ such x.

Define

$$\boxed{\ln(x) = \int_1^x \frac{1}{t}\, dt.}$$

We have

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \forall\, x > 1.$$

We have from the definition

1) $\ln(xy) = \ln(x) + \ln(y)$

and 2) $\ln(x^r) = r\ln(x)$

$\forall\, x, y \geq 1,\ r \geq 0.$

Extend to $(0,1)$ by defining

$$\ln(x) = -\int_x^1 \frac{1}{t}\,dt$$

$$\left(= \int_1^x \frac{1}{t}\,dt \right).$$

All previous properties apply except we can now take

$x, y > 0$, $r \in \mathbb{R}$.

Then $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$\forall\ x > 0$ by the

fundamental theorem

$\Rightarrow$ ln is always increasing.

On $[1,2]$,

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \geq \int_1^x \frac{1}{2}\,dt$$

$$\geq \int_1^2 \frac{1}{2}\,dt$$

$$= 1/2$$

Similarly, on $[2,3]$,

$\ln(x) \geq 1/3$ and on

$[3,4]$, $\ln(x) \geq 1/4$.

Therefore,

$$\ln(4) = \int_1^4 \frac{1}{t}\,dt$$

$$= \int_1^2 \frac{1}{t}\,dt + \int_2^3 \frac{1}{t}\,dt + \int_3^4 \frac{1}{t}\,dt$$

$$\geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1.$$

So $\ln(1) = 0$, $\ln(4) > 1$,

$\ln$ is always increasing,

and $\ln$ is continuous
(in fact, differentiable)

on $(0, \infty)$.

By the intermediate

value theorem, $\exists$ (unique)

$e \in (0, 4)$ with

$\ln(e) = 1$.

$\boxed{\text{Define } f(x) = e^x \text{ to be the inverse function of } \ln.}$

$e^x : \mathbb{R} \to \mathbb{R}.$

We then know

$$\ln(e^x) = x \ln(e) = x,$$

so we have chosen the correct inverse!

We want to compute

$$\frac{d}{dx}(e^x).$$

Let $f(x) = e^x$, $f^{-1}(x) = \ln(x)$

$$f(f^{-1}(x)) = x.$$

Differentiate both sides.

By the chain rule,

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1$$

$$f^{-1}(x) = \ln(x), \quad \frac{d}{dx}(f^{-1}(x)) = \frac{1}{x},$$

so

$$f'(\ln(x)) = x.$$

Let $x = e^y$. Then

$$\boxed{f'(y) = e^y = f(y).}$$

Uniform Convergence and Differentiation

Example 1: $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = x^n$$

On $[0,1]$, $f_n \to f$ pointwise

where $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$

f_n is differentiable

$\forall\ n \in \mathbb{N},\ x \in [0,1]$,

but f is not even

continuous at $x=1$!

Therefore, pointwise convergence of differentiable functions does not guarantee differentiability of the limit.

Example 2: $f_n : [-1,1] \to \mathbb{R}$

$$f_n(x) = x^{1 + \frac{1}{2n-1}}$$

Claim: $f_n \to f$

where $f(x) = |x|$

uniformly on $[-1,1]$.

We know that

$$f_n'(x) = \left(1 + \frac{1}{2n-1}\right) x^{\frac{1}{2n-1}}$$

valid everywhere in $[-1,1]$.

In particular, f_n is differentiable at $x=0$ $\forall\, n \in \mathbb{N}$. But f is not differentiable at $x=0$!